INTRINSIC KNOTTING AND LINKING OF ALMOST COMPLETE PARTITE GRAPHS

THOMAS W. MATTMAN, RYAN OTTMAN, AND MATT RODRIGUES

Abstract. We classify graphs that are 0, 1, or 2 edges short of being complete partite graphs with respect to intrinsic linking and intrinsic knotting. In addition, we classify intrinsic knotting of graphs on 8 vertices. For graphs in these families, we verify a conjecture presented in Adams' *The Knot Book*: If a vertex is removed from an intrinsically knotted graph, one obtains an intrinsically linked graph.

1. Introduction

We say that a graph is intrinsically knotted (respectively, linked) if every tame embedding of the graph in $\mathbb{R}^3$ contains a non-trivially knotted cycle (respectively, pair of non-trivially linked cycles). Robertson, Seymour, and Thomas [9] demonstrated that intrinsic linking is determined by the seven Petersen graphs. A graph is intrinsically linked if and only if it is or contains one of the seven as a minor. They also showed [8] that a similar, finite list of graphs exists for the intrinsic knotting property. However, determining this list remains difficult.

It is known [2, 4, 6, 7] that K_7 and $K_{3,3,1,1}$ along with any graph obtained from these two by triangle-Y exchanges is minor minimal with respect to intrinsic knotting. Recently Foisy [5] has added a new minor minimal graph to the list. Foisy's example is particularly interesting from our perspective as it provides a counterexample to the "unsolved question" posed in Adams' [1] book: *Is it true that if G [a graph] is intrinsically knotted and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked?*

While Adams' conjecture is not true in general, it does appear to hold for a wide array of graphs, particularly graphs that are "almost" complete. We will say a graph is k**-deficient** if it is a complete or complete partite graph with k edges removed. In the current article we verify Adams' conjecture for 0-, 1-, and 2-deficient graphs. This can be seen as the first few steps in a project to find a counterexample to Adams' conjecture of minimum deficiency. However, as Foisy's counterexample is 13-deficient, there is a long way to go in this program.

More promising is the search for a counterexample on a minimum number of vertices. Since K_7 is a minor minimal intrinsically knotted graph, it is the only graph on 7 or fewer vertices that is intrinsically knotted. In the current paper, we show that there are twenty intrinsically knotted graphs on 8 vertices. These all satisfy Adams' conjecture, so a minimal counterexample to the conjecture must have between 9 vertices and the 13 of Foisy's graph.

In classifying intrinsic knotting of various families, we have made use of the known minor minimal examples derived from K_7 and $K_{3,3,1,1}$ by triangle-Y exchanges. In particular, we include here a table of the 25 graphs obtained from $K_{3,3,1,1}$ ([6] includes the table built on K_7). Note that there are no new examples of minor minimal intrinsically knotted graphs to be found among 0-, 1-, and 2-deficient graphs and graphs on 8 vertices.

The paper is organized as follows. Following this introduction, Section 2 presents some fundamental lemmas and the table of graphs obtained from $K_{3,3,1,1}$ by triangle-Y exchanges. In Sections 3,4, and 5 we investigate 0-, 1-, and

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2-deficient graphs respectively. In each case we classify the graphs with respect to intrinsic knotting and linking and demonstrate that they satisfy Adams' conjecture. In Section 6 we classify intrinsically knotted graphs on 8 vertices and verify that they too satisfy Adams' conjecture. We complete the paper by formulating the analogue to a question Sachs asked about intrinsic linking. We show that if $5 \leq n \leq 8$, a graph on n vertices that is not intrinsically knotted will have at most $5n - 15$ edges and ask if this is true more generally:
Question: Is there a graph on n vertices that is not intrinsically knotted and has more than $5n - 15$ edges?

2. Lemmas and graphs derived from $K_{3,3,1,1}$

In this section we present some useful lemmas as well as a table of graphs derived from $K_{3,3,1,1}$ by triangle-Y exchanges. Let us begin with some notation. We will use $K_{a_1,a_2,\ldots,a_p}$ to denote a complete partite graph with p parts containing respectively $a_1, a_2, \ldots, a_p$ vertices. Permuting the a_i results in the same graph. We will usually write the parts in descending order: $a_1 \geq a_2 \geq \ldots \geq a_p$. The complete graph on n vertices is denoted K_n.

Recall that a **minor** of a graph G is the resultant graph after performing a finite number of vertex or edge deletions and edge contractions on G. An example of an **edge contraction** is shown in figure 1 (moving from left to right in the figure). Moving from right to left in the figure is known as **splitting a vertex**.

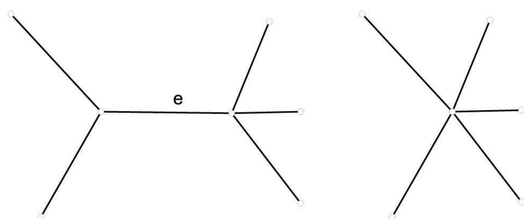

Figure 1. An edge contraction of edge e

We say a graph G is **minor minimal** with respect to a property if G exhibits the property but no minor of G does. For example, the seven Petersen graphs are the only graphs minor minimal with respect to intrinsic linking. An intrinsically linked graph must either be a Petersen graph or contain one as a minor.

The analogous list of graphs for intrinsic knotting is incomplete at this time. We do know that the list is finite [8] and if G is in the list, so is any graph obtained from G by triangle-Y exchanges [7]. To perform a **triangle-Y** exchange on a graph, find 3 vertices that are all connected to one another, delete the three edges between them, and replace with a single vertex connected to all 3 vertices of the triangle. Kohara and Suzuki [6] have provided a list of the 13 graphs arising from K_7 by triangle-Y exchanges. They state that these graphs are minor-minimal with respect to intrinsic knotting and remark that there are 25 graphs that can be constructed from $K_{3,3,1,1}$ by triangle-Y exchanges. We present these graphs in figure 2. Since Foisy [4] showed that $K_{3,3,1,1}$ is intrinsically linked it follows [6] that all 26 graphs in the figure are minor minimal with respect to intrinsic knotting.

The graphs in figure 2 are organized into rows having equal numbers of vertices. The arrows indicate that a triangle-Y exchange has occurred. A dashed arrow means that the triangle-Y exchange will result in a different projection of the graph than that shown in the figure. Note that graphs B_{13}, C_{13}, and A_{14} contain no triangles. Most graphs in the figure can be shown to be different from the others by examining the set of vertex degrees. The exceptions are the pairs A_{11} and B_{11}, E_{11} and G_{11}, A_{12} and B_{12}, and A_{13} and B_{13}.

We can distinguish the members of these pairs as follows. B_{11} contains a triangle of degree 5 vertices while A_{11} does not. G_{11} contains a degree 3 vertex that is connected only to degree 4 vertices; E_{11} contains no such degree 3 vertex. A_{12} contains a 5,5,4 triangle. The two degree 5 vertices in B_{12} are connected to no common degree 4 vertex so such a triangle is not possible. A_{13} contains a triangle; no triangles exist in B_{13}

These lists of known intrinsically linked or knotted graphs will be one of the two main techniques used in our classifications. If we know that a graph G contains a linked (knotted) minor, then G must also be linked (knotted).

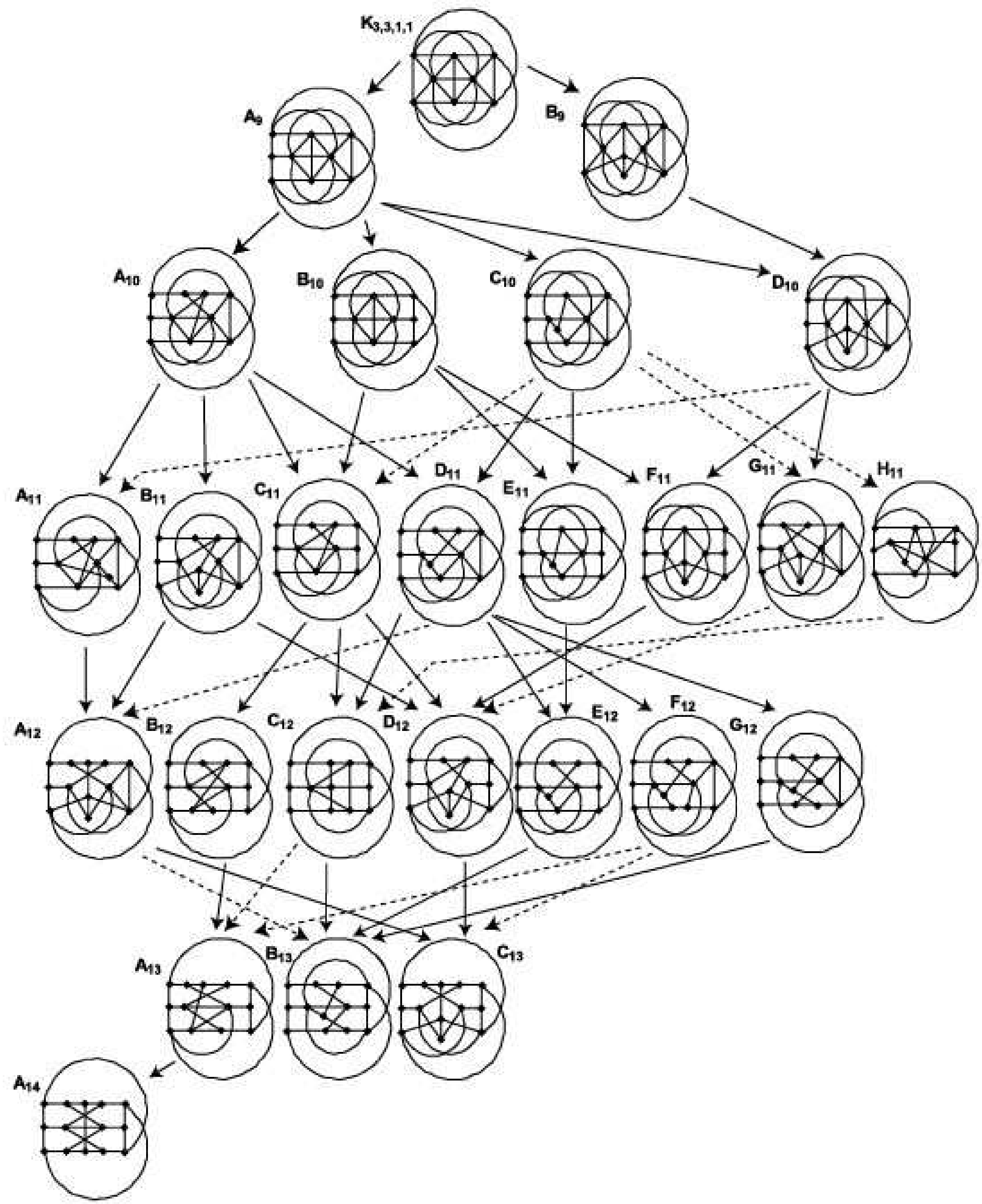

FIGURE 2. Graphs obtained from $K_{3,3,1,1}$ by triangle-Y exchanges

Conversely, if we can realize G as a minor of an unlinked (unknotted) graph, we deduce that G must also be unlinked (unknotted). A useful lemma in this regard shows how we can combine parts of a k-deficient graph.

Lemma 1. *$K_{n_1+n_2,n_3,\dots,n_p}$ is a minor of $K_{n_1,n_2,\dots,n_p}$. Similarly, $K_{n_1+n_2,n_3,\dots,n_p}-k$ edges is a minor of $K_{n_1,n_2,\dots,n_p}-k$ edges.*

Proof: Combining the n_1 and n_2 parts only involves removing edges between vertices in the n_1 part and the n_2 part. Recall that with complete partite graphs, the ordering of the subscripts is not important, so this lemma implies that we can combine any 2 parts to get a minor of the original graph.

Now, suppose we have a complete partite graph with k edges removed and we combine two parts. The result would be a complete partite graph with 1 fewer part and at most k edges removed.

Furthermore, if there are m edges missing between parts n_1 and n_2, we can see that $K_{n_1+n_2,n_3,\dots,n_k}-(k-m)$ edges is a minor of $K_{n_1,n_2,\dots,n_k}-k$ edges. □

The other main technique we will use in our classifications is based on a lemma for intrinsic linking due to Sachs [10] and an analogous result for intrinsic knotting proved by Fleming [3]. Let $G+H$ denote the suspension of graphs G and H, i.e., the graph obtained by taking the union of G and H and adding an edge between each vertex of G and each vertex of H.

Lemma 2 ([10])**.** *The graph $G+K_1$ is intrinsically linked if and only if G is non-planar*

Corollary 3. *If a vertex is deleted from a graph H and the result is a planar graph, then H is not intrinsically linked.*

Proof:

If the deleted vertex was connected to every other vertex, then by lemma 2, H is not intrinsically linked. If the deleted vertex was not connected to every other vertex, then H is a minor of a graph that is not intrinsically linked by lemma 2. □

Lemma 4 ([3])**.** *The graph $G+K_2$ is intrinsically knotted if and only if G is non-planar*

Corollary 5. *If two vertices are deleted from a graph H and the result is a planar graph, then H is not intrinsically knotted.*

Note that if we use lemma 4 to argue that a graph $H=G+K_2$ is intrinsically linked, then, G must contain K_5 or $K_{3,3}$ as a minor. It follows that H contains K_7 or $K_{3,3,1,1}$ as a minor. As our only other technique relies on the family of graphs obtained by triangle-Y exchanges from K_7 and $K_{3,3,1,1}$, our methods cannot possibly add to the list of known minor-minimal graphs. In other words, every intrinsically knotted graph that is 0-, 1-, or 2-deficient or a graph on 8 vertices necessarily is or has as a minor one of the minor minimal graphs obtained from K_7 or $K_{3,3,1,1}$ by triangle-Y exchanges.

3. Complete and complete partite graphs

In this section we classify complete graphs and complete partite graphs with respect to intrinsic linking. The classification of these graphs with respect to intrinsic knotting is due to Fleming [3]. We use these classifications to prove that Adams' conjecture holds for this class of graphs.

Theorem 6. *The complete k-partite graphs are classified with respect to intrinsic linking according to Table 1.*

k	1	2	3	4	5	≥ 6
linked	6	4,4	3,3,1 4,2,2	2,2,2,2 3,2,1,1	2,2,1,1,1 3,1,1,1,1	All
not linked	5	n,3	3,2,2 n,2,1	2,2,2,1 n,1,1,1	2,1,1,1,1	None

Table 1. Intrinsic linking of complete k-partite graphs.

Remark: The n in the notation $K_{n,3}$ indicates that the property holds for any number of vertices in that part, i.e., none of the graphs $K_{1,3}$, $K_{2,3}$, $K_{3,3}$, ... are intrinsically linked. For each k, the table includes minimal examples of intrinsically linked complete k-partite graphs and maximal graphs which are not intrinsically linked. For example, any complete 2-partite graph which contains $K_{4,4}$ as a minor is intrinsically linked. On the other hand, any complete 2-partite graph which is a minor of a $K_{n,3}$ is not linked. Thus, $K_{l,m}$ is intrinsically linked if and only if $l \geq 4$ and $m \geq 4$.

Proof:

Let us demonstrate that the graphs labeled "linked" in Table 1 are in fact intrinsically linked.

Conway and Gordon [2] proved that K_6 is linked. Any k-partite graph with $k \geq 6$ contains $K_6 = K_{1,1,1,1,1,1}$ as a minor and is therefore linked. For the remaining k, we appeal to work of Robertson, Seymour, and Thomas [9] who showed that a graph is intrinsically linked if and only if it is or contains as a minor one of the seven graphs in the Petersen family. In particular, $K_{4,4}$ with one edge removed and $K_{3,3,1}$ are both in this family. By combining parts, we see that, for $2 \leq k \leq 5$, one of these two is a minor of each of the "linked" graphs in Table 1.

For each of the "not linked" examples in Table 1 which involve a part with a single vertex, the corresponding graph obtained by removing that vertex is planar. Therefore, by lemma 4, these graphs are not intrinsically linked.

Since a cycle requires at least three vertices, K_5 has no disjoint pair of cycles and is therefore not linked. The remaining "not linked" graphs in Table 1, $K_{n,3}$ and $K_{3,2,2}$, are, respectively, minors of the unlinked graphs $K_{n,2,1}$ and $K_{2,2,2,1}$. □

Theorem 7. *If G is an intrinsically knotted complete partite graph, and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked.*

Proof: For the reader's convenience, we reproduce Fleming's [3] classification of knotted partite graphs as Table 2 below.

k	1	2	3	4	5	6	≥ 7
knotted	7	5,5	3,3,3 4,3,2 4,4,1	3,2,2,2 4,2,2,1 3,3,2,1 3,3,1,1	2,2,2,2,1 3,2,2,1,1 3,2,1,1,1	2,2,1,1,1,1 3,1,1,1,1,1	All
not knotted	6	4,4	3,3,2 n,2,2 n,3,1	2,2,2,2 4,2,1,1 3,2,2,1 n,2,1,1 n,1,1,1	2,2,2,1,1 2,2,1,1,1 n,1,1,1,1	2,1,1,1,1,1	None

TABLE 2. Intrinsic knotting of k-partite graphs.

It suffices to verify the theorem for minimal examples of knotted k-partite graphs, $k = 1, 2, 3, \ldots$.

- $k = 1$**:** A complete graph K_n is intrinsically knotted iff $n \geq 7$. Removing a vertex from K_7 produces K_6 which is intrinsically linked.
- $k = 2$**:** $K_{5,5}$ is the minor-minimal knotted 2-partite graph. (Note that $K_{n,4}$ is not intrinsically knotted for $n \geq 5$. This is implicit in [3].) Removing a vertex from $K_{5,5}$ yields $K_{5,4}$ which is intrinsically linked.
- $k = 3$**:** There are three minor-minimal knotted 3-partite graphs: $K_{3,3,3}$, $K_{4,3,2}$, and $K_{4,4,1}$. Removing a vertex from one of these results in one of the following linked graphs: $K_{4,4}$, $K_{3,3,2}$, $K_{4,2,2}$, or $K_{4,3,1}$.
- $k = 4$**:** Here the minimal knotted graphs are $K_{3,2,2,2}$, $K_{3,3,1,1}$ and $K_{4,2,2,1}$ (The graph $K_{3,3,2,1}$ listed by Fleming [3] is redundant as it includes $K_{3,3,1,1}$ as a minor.) Removing a vertex from any of these we obtain one of the linked graphs $K_{3,3,1}$, $K_{4,2,2}$, $K_{2,2,2,2}$, $K_{3,2,1,1}$, $K_{3,2,2,1}$, or $K_{4,2,1,1}$.

$k = 5$**:** In this case we must check the knotted graphs $K_{2,2,2,2,1}$ and $K_{3,2,1,1,1}$. (Fleming's [3] $K_{3,2,2,1,1}$ is redundant.) Taking a vertex from either of these results in a linked graph: $K_{2,2,2,2}$, $K_{3,2,1,1}$, $K_{2,2,1,1,1}$, $K_{2,2,2,1,1}$, or $K_{3,1,1,1,1}$.

$k = 6$**:** Here there are two knotted graphs: $K_{2,2,1,1,1,1}$ and $K_{3,1,1,1,1,1}$. After a vertex is deleted, we're left with one of these linked graphs: $K_{2,2,1,1,1}$, $K_{3,1,1,1,1}$, or $K_{2,1,1,1,1,1}$.

$k \geq 7$**:** All such graphs are intrinsically knotted. On removing a vertex, we obtain an l-partite graph where $l \geq 6$. All such graphs are intrinsically linked.

□

4. 1-deficient graphs

In this section we classify 1-deficient graphs with respect to intrinsic linking and knotting. We use the classification to prove Adams' conjecture for this family of graphs.

Notation: Often, we will have to talk about a particular vertex or part. We will refer to parts alphabetically with capital letters and vertices of those parts with lower case letters. For example, in $K_{4,3,1}$, we will call the part with 4 vertices part A, the part with 3 vertices part B, and the part with 1 vertex part C. An edge between parts A and C would be labeled (a,c).

4.1. **Intrinsic linking.**

Theorem 8. *The 1-deficient graphs are classified with respect to intrinsic linking according to Table 3.*

k	1	2	3	4	5	6	≥ 7
linked	7-e	4,4-e	4,3,1-e 3,3,2-e 4,2,2-e	2,2,2,2-e 3,2,1,1-(b,c) 4,2,1,1-e 3,3,1,1-e 3,2,2,1-e	2,2,1,1,1-(b,c) 3,1,1,1,1-(b,c) 4,1,1,1,1-e 3,2,1,1,1-e 2,2,2,1,1-e	2,1,1,1,1,1-e	All
not linked	6-e	n,3-e	3,2,2-e n,2,1-e 3,3,1-e	2,2,2,1-e n,1,1,1-e 3,2,1,1-(a,b) 3,2,1,1-(a,c) 3,2,1,1-(c,d)	2,2,1,1,1-(a,b) 2,2,1,1,1-(c,d) 3,1,1,1,1-(a,b) 2,1,1,1,1-e	1,1,1,1,1,1-e	None

Table 3. Intrinsic Linking of 1 Deficient Graphs.

Remark:

Note that most of these graphs have different types of edges. For some graphs, removal of one edge (e.g., (a,b)) will result in a non-linked graph while the removal of a different edge (e.g., (b,c)) will result in a linked graph. However, in many cases, removal of any edge will result in the same categorization. In such cases we simply write "-e". For example, no matter which edge we remove from $K_{3,2,2}$, we will obtain a graph that is not intrinsically linked.

Proof:

<u>Linked:</u>

Let us demonstrate that the graphs labeled "linked" in Table 3 are in fact intrinsically linked.

K_7-e has K_6 as a minor. To see this, simply delete a vertex that the removed edge was attached to. Any 1-deficient graph with 7 or more parts will contain K_7-e (or, equivalently, $K_{1,1,1,1,1,1,1}$-e).

Recall that $K_{4,4}$-e is a Petersen graph, and therefore is intrinsically linked. Note that by lemma 1, $K_{4,3,1}$-e, $K_{4,2,2}$-e, $K_{2,2,2,2}$-e, $K_{4,2,1,1}$-e, $K_{3,3,1,1}$-e, $K_{3,2,2,1}$-e, $K_{4,1,1,1,1}$-e, $K_{3,2,1,1,1}$-e, and $K_{2,2,2,1,1}$-e all contain $K_{4,4}$-e as a minor, and are therefore all intrinsically linked.

$K_{3,3,2}$-e has 2 cases: $K_{3,3,2}$-(a,b) and $K_{3,3,2}$-(b,c); both are intrinsically linked. For $K_{3,3,2}$-(a,b), contract the edge between vertex b and c to get $K_{3,2,1,1}$. For $K_{3,3,2}$-(b,c), simply delete vertex c for $K_{3,3,1}$. So in either case, $K_{3,3,2}$-e is intrinsically linked.

By lemma 1, $K_{3,2,1,1}$-(b,c), $K_{2,2,1,1,1}$-(b,c), and $K_{3,1,1,1,1}$-(b,c) all contain $K_{3,3,1}$ as a minor and are therefore intrinsically linked.

There are 2 cases of $K_{2,1,1,1,1,1}$-e: $K_{2,1,1,1,1,1}$-(a,b) and $K_{2,1,1,1,1,1}$-(b,c). For $K_{2,1,1,1,1,1}$-(a,b), simply delete a to get K_6, and notice that $K_{2,1,1,1,1,1}$-(b,c) is equivalent to $K_{2,2,1,1,1}$. So both cases are intrinsically linked.

<u>Not Linked:</u>

K_6 (or $K_{1,1,1,1,1,1}$) and $K_{3,3,1}$ are Petersen graphs and therefore minor minimal with respect to intrinsic linking [9], so K_6-e and $K_{3,3,1}$-e are not intrinsically linked by definition of minor minimal.

$K_{n,3}$-e, $K_{3,2,2}$-e, $K_{n,2,1}$-e, $K_{2,2,2,1}$-e, $K_{n,1,1,1}$-e, and $K_{2,1,1,1,1}$-e are all not intrinsically linked before the edge is removed; so, naturally, we can remove an edge and still have a non-intrinsically linked graph.

$K_{3,2,1,1}$-(c,d) is equivalent to $K_{3,2,2}$ which is not intrinsically linked.

The remaining graphs all have a vertex that is connected to every other vertex, and the removal of that vertex results in a planar graph. So, by lemma 4, none are intrinsically linked. □

k	1	2	3	4	5	6	7	≥ 8
knotted	8-e	5,5-e	3,3,3-e 4,3,2-e 4,4,1-e	3,2,2,2-e 4,2,2,1-e 3,3,2,1-e 4,3,1,1-e	2,2,2,2,1-e 3,2,1,1,1-(b,c) 4,2,1,1,1-e 3,3,1,1,1-e 3,2,2,1,1-e	2,2,1,1,1,1-(b,c) 3,1,1,1,1,1-(b,c) 3,2,1,1,1,1-e 2,2,2,1,1,1-e 4,1,1,1,1,1-e	2,1,1,1,1,1,1-e	All
not knotted	7-e	n,4-e	3,3,2-e n,2,2-e n,3,1-e	3,3,1,1-e 2,2,2,2-e 3,2,2,1-e n,2,1,1-e	3,2,1,1,1-(c,d) 3,2,1,1,1-(a,b) 3,2,1,1,1-(a,c) 2,2,2,1,1-e n,1,1,1,1-e	2,2,1,1,1,1-(a,b) 2,2,1,1,1,1-(c,d) 3,1,1,1,1,1-(a,b) 2,1,1,1,1,1-e	1,1,1,1,1,1,1-e	None

Table 4. Intrinsic knotting of 1 deficient graphs.

4.2. **Intrinsic knotting.**

Theorem 9. *The 1-deficient graphs are classified with respect to intrinsic knotting according to Table 4.*

Proof:

<u>Knotted:</u>

Let us demonstrate that the graphs labeled "knotted" in Table 4 are in fact intrinsically knotted.

K_8-e is has K_7 as a minor; to see this, simply delete a vertex that the removed edge was connected to. All 1-deficient graphs with 8 or more parts will contain $K_{1,1,1,1,1,1,1,1}$-e = K_8-e.

$K_{5,5} - e$ is intrinsically knotted as proved in [11]. It is also helpful to see that it is an expansion of H_9 from [6], a fact pointed out in [4].

$K_{3,3,3} - e$ is intrinsically knotted. As shown in figure 3, several edges are added to H_9 to get $K_{3,3,3}$. H_9 was shown to be intrinsically knotted in [6]. So, if we simply add 1 fewer of those edges, we arrive at $K_{3,3,3} - e$ and our graph is still intrinsically knotted. Notice that any two edges in $K_{3,3,3}$ are equivalent to one another.

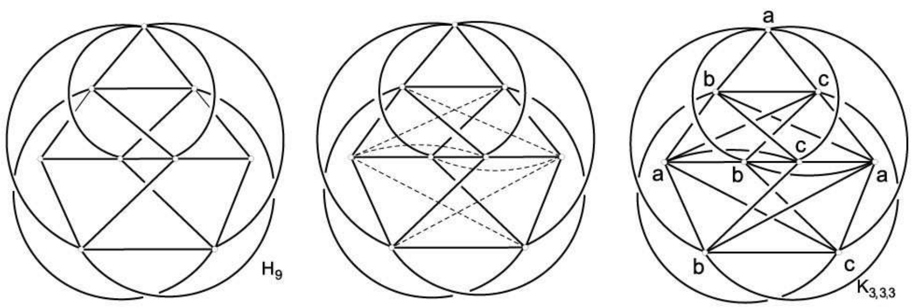

FIGURE 3. H_9 is a Minor of $K_{3,3,3}$

By lemma 1, $K_{3,3,2,1}$-e, $K_{3,3,1,1,1}$-e, $K_{3,2,2,1,1}$-e, $K_{3,2,1,1,1,1}$-e, and $K_{2,2,2,1,1,1}$-e all have $K_{3,3,3}$-e as a minor, and are therefore all intrinsically knotted.

$K_{4,3,2} - e$ is intrinsically knotted, there are 3 cases.

Case 1: $K_{4,3,2} - (a, b)$. $K_{4,3,2}$ is intrinsically knotted as it contains $K_{3,3,1,1}$ as a minor (see figure 4). Notice that if we add one fewer edge, we get $K_{4,3,2} - (a, b)$.

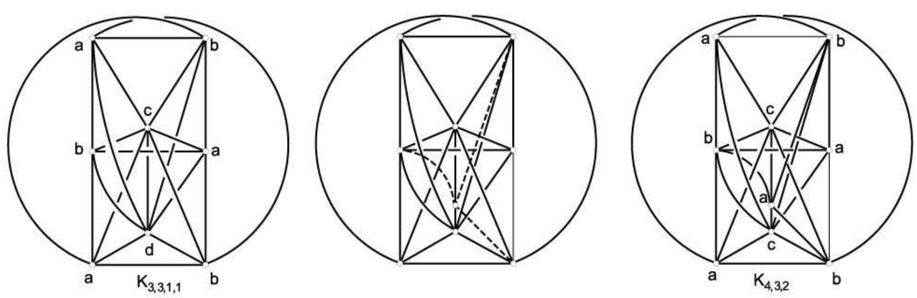

FIGURE 4. $K_{3,3,1,1}$ is a Minor of $K_{4,3,2}$

Case 2: $K_{4,3,2} - (b, c)$. In figure 5, $K_{4,3,2}$ is shown to contain H_9 from [6]. If we leave out edge (b′,c′), we see that $K_{4,3,2} - (b, c)$ contains H_9 as a minor; therefore it is intrinsically knotted.

Case 3: $K_{4,3,2} - (a, c)$. The same as case 2, except we leave out edge (a′,c′).

By lemma 1, $K_{3,2,2,2}$-e, $K_{4,2,2,1}$-e, $K_{4,3,1,1}$-e, $K_{2,2,2,2,1}$-e, $K_{4,2,1,1,1}$-e, and $K_{4,1,1,1,1,1}$-e all contain $K_{4,3,2}$-e as a minor, and are therefore intrinsically knotted.

$K_{4,4,1} - e$ is intrinsically knotted; there are 2 cases: $K_{4,4,1} - (a, b)$ and $K_{4,4,1} - (a, c)$.

As shown in figure 6, $K_{4,4,1}$ contains H_9 as a minor. If we refrain from adding either (a′,b′) or (a′,c′), we can see that both cases of $K_{4,4,1}$-e have H_9 as a minor, and are therefore intrinsically knotted.

By (the proof of) lemma 1, if we combine the parts of a missing edge of a 1-deficient graph, we will get a complete graph. If we combine parts B and C of $K_{3,2,1,1,1}$-(b,c), $K_{2,2,1,1,1,1}$-(b,c), or $K_{3,1,1,1,1,1}$-(b,c), we will get $K_{3,3,1,1}$ or $K_{3,2,1,1,1}$. So these three graphs are intrinsically knotted.

$K_{2,1,1,1,1,1,1}$-e has 2 cases: $K_{2,1,1,1,1,1,1}$-(a,b) and $K_{2,1,1,1,1,1,1}$-(b,c). For $K_{2,1,1,1,1,1,1}$-(a,b), simply delete vertex a to get K_7. $K_{2,1,1,1,1,1,1}$-(b,c) is equivalent to $K_{2,2,1,1,1,1}$. So both cases are intrinsically knotted.

<u>Not Knotted:</u>

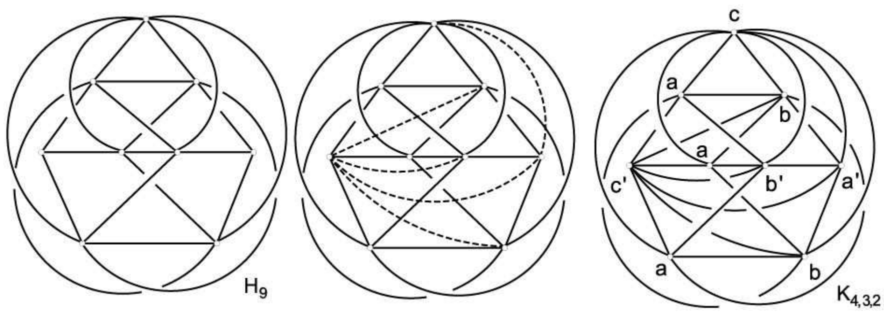

FIGURE 5. H_9 is a Minor of $K_{4,3,2}$

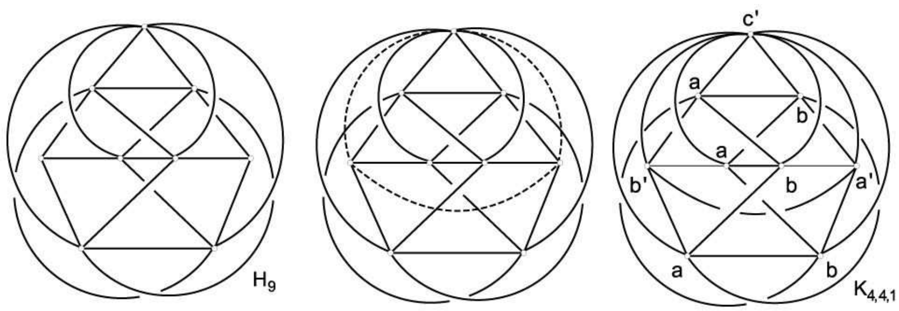

FIGURE 6. H_9 is a Minor of $K_{4,4,1}$

K_7 was shown to be minor minimally intrinsically knotted in [2] and $K_{3,3,1,1}$ was shown to be minor minimally intrinsically knotted in [4], so K_7-e (or $K_{1,1,1,1,1,1,1}$-e) and $K_{3,3,1,1}$-e are not intrinsically knotted.

In Fleming's paper [3], he showed that $K_{4,4}$ is not intrinsically knotted. In fact, his proof really showed that $K_{n,4}$ is not intrinsically knotted, so, naturally, $K_{n,4}$-e is not intrinsically knotted either. Similarly, $K_{3,3,2}$-e, $K_{n,2,2}$-e, $K_{n,3,1}$-e, $K_{2,2,2,2}$-e, $K_{3,2,2,1}$-e, $K_{n,2,1,1}$-e, $K_{2,2,2,1,1}$-e, $K_{n,1,1,1,1}$-e, and $K_{2,1,1,1,1,1}$-e are not intrinsically knotted.

$K_{3,2,1,1,1}$-(c,d) is equivalent to $K_{3,2,2,1}$.

The remaining 5 graphs each have 2 vertices connected to every other vertex. If those 2 vertices are deleted, the resulting graph is planar, so by lemma 4, they are not intrinsically knotted. □

4.3. **Proof of Adams' conjecture for 1-deficient graphs.**

Theorem 10. *If G is a 1-deficient graph, and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked.*

Proof:

It suffices to verify the theorem for minimal examples of knotted 1-deficient graphs.

- **$k = 1$:** K_8-e is intrinsically knotted. If we remove a vertex, we get either K_7 or K_7-e, both of which are intrinsically linked.
- **$k = 2$:** $K_{5,5}$-e is intrinsically knotted. If we remove a vertex, we get either $K_{5,4}$-e or $K_{5,4}$, both of which are intrinsically linked.

$k = 3$: $K_{3,3,3}$-e, $K_{4,3,2}$-e, and $K_{4,4,1}$-e are intrinsically knotted. If we remove a vertex from any of these graphs we will get $K_{3,3,2}$-e, $K_{4,2,2}$-e, $K_{4,3,1}$-e, $K_{4,4}$-e, or one of these without the removed edge. In any case, the result is intrinsically linked.

$k = 4$: $K_{3,2,2,2}$-e, $K_{4,2,2,1}$-e, $K_{3,3,2,1}$-e, and $K_{4,3,1,1}$-e are intrinsically knotted. If we remove a vertex from any of these graphs we will get $K_{4,3,1}$-e, $K_{3,3,2}$-e, $K_{4,2,2}$-e, $K_{2,2,2,2}$-e, $K_{4,2,1,1}$-e, $K_{3,3,1,1}$-e, $K_{3,2,2,1}$-e, or one of these without the edge removed. In all cases, the result is intrinsically linked.

$k = 5$: $K_{2,2,2,2,1}$-e, $K_{3,2,1,1,1}$-(b,c), $K_{4,2,1,1,1}$-e, $K_{3,3,1,1,1}$-e, $K_{3,2,2,1,1}$-e are intrinsically knotted. If we remove a vertex, we will get $K_{2,2,2,2}$-e, $K_{3,2,1,1}$-(b,c), $K_{4,2,1,1}$-e, $K_{3,3,1,1}$-e, $K_{3,2,2,1}$-e, $K_{2,2,1,1,1}$-(b,c), $K_{3,1,1,1,1}$-(b,c), $K_{4,1,1,1,1}$-e, $K_{3,2,1,1,1}$-e, $K_{2,2,2,1,1}$-e, or one of these without the removed edge. In all cases, the result is intrinsically linked.

$k = 6$: $K_{2,2,1,1,1,1}$-(b,c), $K_{3,1,1,1,1,1}$-(b,c), $K_{3,2,1,1,1,1}$-e, $K_{2,2,2,1,1,1}$-e, and $K_{4,1,1,1,1,1}$-e are intrinsically knotted. If we remove a vertex, we will get $K_{2,2,1,1,1}$-(b,c), $K_{3,1,1,1,1}$-(b,c), $K_{4,1,1,1,1}$-e, $K_{3,2,1,1,1}$-e, $K_{2,2,2,1,1}$-e, $K_{2,1,1,1,1,1}$-(b,c), $K_{3,1,1,1,1,1}$-e, $K_{2,2,1,1,1,1}$-e, or one of these without the removed edge. In all cases, the result is intrinsically linked.

$k = 7$: $K_{2,1,1,1,1,1,1}$-e is intrinsically knotted. If we remove a vertex, we will get $K_{2,1,1,1,1,1}$-e, $K_{1,1,1,1,1,1,1}$-e, or one of these without the removed edge. All will be intrinsically linked.

$k \geq 8$: All 1-deficient graphs with 8 or more parts are intrinsically knotted. If we delete a vertex we will have a 1-deficient or complete partite graph with at least 7 parts, all of which are intrinsically linked.

5. 2-deficient graphs

In this section we classify 2-deficient graphs with respect to intrinsic linking and knotting. We also show that Adams' conjecture holds for these graphs.

Notation: We will expand the notation that we created in the last section by adding subscripts to the vertices. For example, if we are removing two edges between parts A and B of $K_{4,3,1}$, there are 3 cases to be considered: deleting two edges that share a vertex from part A ($K_{4,3,1}$-$\{(a_1, b_1), (a_1, b_2)\}$), deleting two edges that share a vertex from part B ($K_{4,3,1}$-$\{(a_1, b_1), (a_2, b_1)\}$), and deleting two edges that share no vertices ($K_{4,3,1}$-$\{(a_1, b_1), (a_2, b_2)\}$). Also, in some cases, if we delete a particular edge we can then delete any other without affecting the classification of the resulting graph. For example, if we delete edge (b,c) from $K_{4,3,1}$, we can then delete any other and still have an intrinsically linked graph; we will denote graphs obtained in this way by $K_{4,3,1}$-$\{(b, c), e\}$. Furthermore, in some cases, we can delete any 2 edges and still have an intrinsically linked graph; for example, K_7 will be intrinsically linked no matter what 2 edges we delete. We denote these graphs K_7-2e.

5.1. Intrinsic linking.

Theorem 11. *The 2-deficient graphs are classified with respect to intrinsic linking according to Tables 5 and 6.*

Proof:

Linked:

K_7-2e is intrinsically linked; there are 2 cases: K_7-$\{(a, b), (b, c)\}$ and K_7-$\{(a, b), (c, d)\}$. For the first case, we can simply delete vertex b to get K_6; for the second case, we can contract edge (a, c) to get K_6. Notice that any 2-deficient graph with 7 or more parts will have K_7-2e as a minor, and will therefore be intrinsically linked.

$K_{5,4}$-2e is intrinsically linked; there are 3 cases. In each case, one removed edge will be (a_1, b_1). If we simply delete vertex a_1 we will get $K_{4,4}$ or $K_{4,4}$-e.

By lemma 1, $K_{5,3,1}$-2e, $K_{4,4,1}$-2e, $K_{4,3,2}$-2e, $K_{5,2,2}$-2e, $K_{5,2,1,1}$-2e, $K_{4,3,1,1}$-2e, $K_{4,2,2,1}$-2e, $K_{3,3,2,1}$-2e, $K_{5,1,1,1,1}$-2e, $K_{4,2,1,1,1}$-2e, and $K_{3,3,1,1,1}$-2e all have $K_{5,4}$-2e as a minor, and are therefore intrinsically linked.

$K_{3,3,3}$-2e is intrinsically linked. Removing any 2 edges will result in a graph of the form $K_{3,3,3}$-$\{(a, b), e\}$. If we delete vertex a, we will get $K_{3,3,2}$, or $K_{3,3,2}$-e, so $K_{3,3,3}$-2e is intrinsically linked.

$K_{2,2,2,2}$-2e is intrinsically linked. All such graphs are of the form $K_{2,2,2,2}$-$\{(a, b), e\}$. It them follows from lemma 1 that $K_{4,4}$-e is a minor of $K_{2,2,2,2}$-2e.

k	1	2	3	4
linked	7-2e	5,4-2e	4,3,1-$\{(b,c),e\}$	3,2,1,1-$\{(b_1,c),(b_1,d)\}$
			4,3,1-$\{(a_1,b_1),(a_1,b_2)\}$	3,2,1,1-$\{(b_1,c),(b_2,c)\}$
			4,3,1-$\{(a_1,b_1),(a_1,c)\}$	4,2,1,1-$\{(b,c),e\}$
			3,3,2-$\{(a_1,b_1),(a_1,b_2)\}$	4,2,1,1-$\{(c,d),e\}$
			3,3,2-$\{(a_1,c_1),(a_2,c_1)\}$	4,2,1,1-$\{(a_1,b_1),(a_1,b_2)\}$
			3,3,2-$\{(a_1,c_1),(b_1,c_1)\}$	4,2,1,1-$\{(a_1,b_1),(a_1,c)\}$
			3,3,2-$\{(a_1,b_1),(b_2,c_1)\}$	4,2,1,1-$\{(a_1,c),(a_1,d)\}$
			3,3,2-$\{(a_1,c_1),(b_1,c_2)\}$	3,3,1,1-$\{(b,c),e\}$
			3,3,2-$\{(a_1,c_1),(a_1,c_2)\}$	3,3,1,1-$\{(c,d),e\}$
			4,2,2-$\{(b,c),e\}$	3,3,1,1-$\{(a_1,b_1),(a_1,b_2)\}$
			4,2,2-$\{(a_1,b_1),(a_1,b_2)\}$	3,2,2,1-$\{(b,c),e\}$
			5,3,1-2e	3,2,2,1-$\{(c,d),e\}$
			4,4,1-2e	3,2,2,1-$\{(a,d),e\}$
			4,3,2-2e	3,2,2,1-$\{(a_1,b_1),(a_1,b_2)\}$
			3,3,3-2e	3,2,2,1-$\{(a_1,b_1),(a_2,b_1)\}$
			5,2,2-2e	3,2,2,1-$\{(a_1,b_1),(a_2,c_1)\}$
				2,2,2,2-2e
				5,2,1,1-2e
				4,3,1,1-2e
				4,2,2,1-2e
				3,3,2,1-2e
not linked	6-2e	4,4-2e	4,3,1-$\{(a_1,b_1),(a_2,b_1)\}$	3,2,1,1-$\{(a,b),e\}$
		n,3-2e	4,3,1-$\{(a_1,b_1),(a_2,b_2)\}$	3,2,1,1-$\{(a,c),e\}$
			4,3,1-$\{(a_1,c),(a_2,c)\}$	3,2,1,1-$\{(c,d),e\}$
			4,3,1-$\{(a_1,b_1),(a_2,c)\}$	3,2,1,1-$\{(b_1,c),(b_2,d)\}$
			3,3,2-$\{(a_1,b_1),(a_2,b_2)\}$	4,2,1,1-$\{(a_1,b_1),(a_2,b_2)\}$
			3,3,2-$\{(a_1,b_1),(b_1,c_1)\}$	4,2,1,1-$\{(a_1,c),(a_2,c)\}$
			3,3,2-$\{(b_1,c_1),(b_2,c_2)\}$	4,2,1,1-$\{(a_1,b_1),(a_2,b_1)\}$
			4,2,2-$\{(a_1,b_1),(a_2,b_2)\}$	4,2,1,1-$\{(a_1,b_1),(a_2,c)\}$
			4,2,2-$\{(a_1,b_1),(a_2,b_1)\}$	4,2,1,1-$\{(a_1,c),(a_2,d)\}$
			4,2,2-$\{(a_1,b_1),(a_2,c_1)\}$	3,3,1,1-$\{(a_1,b_1),(a_2,b_2)\}$
			4,2,2-$\{(a_1,b_1),(a_1,c_1)\}$	3,2,2,1-$\{(a_1,b_1),(a_2,b_2)\}$
			3,2,2-2e	3,2,2,1-$\{(a_1,b_1),(a_1,c_1)\}$
			n,2,1-2e	2,2,2,1-2e
			3,3,1-2e	n,1,1,1-2e

TABLE 5. Intrinsic Linking of 2 Deficient Graphs

By lemma 1, $K_{2,2,2,1,1}$-2e and $K_{2,2,1,1,1,1}$-2e have $K_{2,2,2,2}$-2e as a minor, and are therefore intrinsically linked.

$K_{3,1,1,1,1,1}$-2e is intrinsically linked. All such graphs are of the form $K_{3,1,1,1,1,1}$-$\{(a,b),e\}$ or $K_{3,1,1,1,1,1}$-$\{(b,c),e\}$. The first case contains $K_{4,4}$-e as a minor by lemma 1, and the second case is equivalent to $K_{3,2,1,1,1}$-e, so $K_{3,1,1,1,1,1}$-2e is intrinsically linked.

By lemma 1, $K_{4,3,1}$-$\{(b,c),e\}$, $K_{4,2,2}$-$\{(b,c),e\}$, $K_{4,2,1,1}$-$\{(b,c),e\}$, $K_{4,2,1,1}$-$\{(c,d),e\}$, $K_{3,3,1,1}$-$\{(b,c),e\}$, $K_{3,2,2,1}$-$\{(b,c),e\}$, $K_{3,2,2,1}$-$\{(a,d),e\}$, $K_{4,1,1,1,1}$-$\{(b,c),e\}$, $K_{3,2,1,1,1}$-$\{(a,c),e\}$, $K_{3,2,1,1,1}$-$\{(b,c),e\}$, and $K_{3,2,1,1,1}$-$\{(c,d),e\}$ all have $K_{4,4}$-e as a minor, and are therefore all intrinsically linked.

$K_{3,3,1,1}$-$\{(c,d),e\}$, and $K_{3,2,2,1}$-$\{(c,d),e\}$ have $K_{3,3,2}$-e as a minor by lemma 1, so they are intrinsically linked.

k	5	6	$\geq$7
linked	2,2,1,1,1-$\{(b_1,c),(b_2,c)\}$	2,1,1,1,1,1-$\{(a_1,b),(a_1,c)\}$	All
	2,2,1,1,1-$\{(a_1,d),(b_1,c)\}$	2,1,1,1,1,1-$\{(a_1,b),(a_2,b)\}$	
	2,2,1,1,1-$\{(b_1,c),(b_1,d)\}$	2,1,1,1,1,1-$\{(a_1,b),(c,d)\}$	
	3,1,1,1,1-$\{(b,c),(c,d)\}$	2,1,1,1,1,1-$\{(b,c),(c,d)\}$	
	4,1,1,1,1-$\{(b,c),e\}$	3,1,1,1,1,1-2e	
	4,1,1,1,1-$\{(a_1,b),(a_1,c)\}$	2,2,1,1,1,1-2e	
	3,2,1,1,1-$\{(a,c),e\}$		
	3,2,1,1,1-$\{(b,c),e\}$		
	3,2,1,1,1-$\{(c,d),e\}$		
	3,2,1,1,1-$\{(a_1,b_1),(a_1,b_2)\}$		
	3,2,1,1,1-$\{(a_1,b_1),(a_2,b_1)\}$		
	2,2,2,1,1-2e		
	5,1,1,1,1-2e		
	4,2,1,1,1-2e		
	3,3,1,1,1-2e		
not linked	2,2,1,1,1-$\{(a,b),e\}$	2,1,1,1,1,1-$\{(a_1,b),(a_2,c)\}$	None
	2,2,1,1,1-$\{(c,d),e\}$	2,1,1,1,1,1-$\{(a_1,b),(b,c)\}$	
	2,2,1,1,1-$\{(b_1,c),(b_2,d)\}$	2,1,1,1,1,1-$\{(b,c),(d,e)\}$	
	2,2,1,1,1-$\{(a_1,c),(b_1,c)\}$	1,1,1,1,1,1-2e	
	3,1,1,1,1-$\{(a,b),e\}$		
	3,1,1,1,1-$\{(b,c),(d,e)\}$		
	4,1,1,1,1-$\{(a_1,b),(a_2,c)\}$		
	4,1,1,1,1-$\{(a_1,b),(a_2,b)\}$		
	3,2,1,1,1-$\{(a_1,b_1),(a_2,b_2)\}$		
	2,1,1,1,1-2e		

TABLE 6. Intrinsic Linking of 2 Deficient Graphs (cont.)

$K_{4,3,1}$-$\{(a_1,b_1),(a_1,b_2)\}$ and $K_{4,3,1}$-$\{(a_1,b_1),(a_1,c)\}$ both have $K_{3,3,1}$ as a minor; simply delete vertex a_1. So both are intrinsically linked.

$K_{4,2,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{4,2,1,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{3,3,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{4,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$, $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, and $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$ all contain $K_{4,3,1}$-$\{(a_1,b_1),(a_1,b_2)\}$ as a minor, so they are all intrinsically linked. (Notice that for the case of $K_{3,2,2,1}$ and $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, once the parts are combined, the labels (a and b) of some of the vertices switch.)

$K_{3,3,2}$-$\{(a_1,b_1),(a_1,b_2)\}$ and $K_{3,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$ have $K_{3,2,1,1}$ as a minor. To see this, simply contract edge (a_1,c_1). So both are intrinsically linked.

By lemma 1, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_1,b_2)\}$ has $K_{3,3,2}$-$\{(a_1,b_1),(a_1,b_2)\}$ as a minor, so it is intrinsically linked.

By lemma 1, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_2,c_1)\}$ has $K_{3,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$ as a minor. Notice that $K_{3,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$ and $K_{3,3,2}$-$\{(b_1,a_1),(a_2,c_1)\}$ are equivalent.

$K_{3,3,2}$-$\{(a_1,c_1),(a_2,c_1)\}$ and $K_{3,3,2}$-$\{(a_1,c_1),(b_1,c_1)\}$ both have $K_{3,3,1}$ as a minor; simply delete vertex c_1. So both are intrinsically linked.

$K_{3,3,2}$-$\{(a_1,c_1),(b_1,c_2)\}$ is intrinsically linked. Delete the remaining edges between c_1 and A and between c_2 and B. The result is $K_{4,4}-e$ where the missing edge is between vertices c_1 and c_2.

$K_{3,3,2}-\{(a_1,c_1),(a_1,c_2)\}$ is intrinsically linked. In figure 7, at left is a Petersen graph, and at right is that same graph with four edges added. As we can see, this is $K_{3,3,2}-\{(a_1,c_1),(a_1,c_2)\}$.

$K_{4,2,2}$-$\{(a_1,b_1),(a_1,b_2)\}$ is intrinsically linked; contract edge (a_1,c_1) to get $K_{3,2,1,1}$

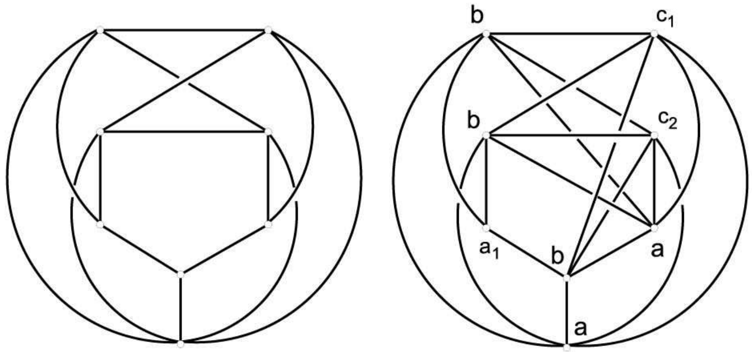

FIGURE 7. $K_{3,3,2} - \{(a_1, c_1), (a_1, c_2)\}$ is Intrinsically Linked

$K_{4,2,1,1}$-$\{(a_1, c), (a_1, d)\}$ has $K_{4,2,2}$-$\{(a_1, b_1), (a_1, b_2)\}$ as a minor by lemma 1.
$K_{3,2,1,1}$-$\{(b_1, c), (b_2, c)\}$ is equivalent to $K_{3,3,1}$ which is intrinsically linked.
$K_{3,2,1,1}$-$\{(b_1, c), (b_1, d)\}$ is a Petersen graph; it is obtained by a triangle-Y exchange of K_6.
$K_{2,2,1,1,1}$-$\{(b_1, c), (b_2, c)\}$ is equivalent to $K_{3,2,1,1}$, so it is intrinsically linked.
$K_{2,2,1,1,1}$-$\{(a_1, d), (b_1, c)\}$ has $K_{3,3,1}$ as a minor by lemma 1.
$K_{2,2,1,1,1}$-$\{(b_1, c), (b_1, d)\}$ has K_6 as a minor; simply contract edge (a_1, b_1).
$K_{3,1,1,1,1}$-$\{(b, c), (c, d)\}$ has $K_{3,3,1}$ as a minor by lemma 1.
$K_{2,1,1,1,1,1}$-$\{(a_1, b), (a_1, c)\}$ has K_6 as a minor; simply delete vertex a_1.
$K_{2,1,1,1,1,1}$-$\{(a_1, b), (a_2, b)\}$ is equivalent to $K_{3,1,1,1,1}$.
$K_{2,1,1,1,1,1}$-$\{(a_1, b), (c, d)\}$ is equivalent to $K_{2,2,1,1,1}$-(a, c)
$K_{2,1,1,1,1,1}$-$\{(b, c), (c, d)\}$ has $K_{3,2,1,1}$ as a minor.

Not Linked:

The following graphs are already known not to be intrinsically linked if we add one more edge, so they are naturally not intrinsically linked: K_6-2e (or $K_{1,1,1,1,1,1}$-2e), $K_{n,3}$-2e, $K_{3,2,2}$-2e, $K_{n,2,1}$-2e, $K_{3,3,1}$-2e, $K_{2,2,2,1}$-2e, $K_{n,1,1,1}$-2e, $K_{3,2,1,1}$-$\{(a, b), e\}$, $K_{3,2,1,1}$-$\{(a, c), e\}$, $K_{3,2,1,1}$-$\{(c, d), e\}$, $K_{2,2,1,1,1}$-$\{(a, b), e\}$, $K_{2,2,1,1,1}$-$\{(c, d), e\}$, $K_{3,1,1,1,1}$-$\{(a, b), e\}$, $K_{2,1,1,1,1}$-2e.

$K_{2,1,1,1,1,1}$-$\{(b, c), (d, e)\}$, $K_{2,1,1,1,1,1}$-$\{(a_1, b), (b, c)\}$, $K_{2,1,1,1,1,1}$-$\{(a_1, b), (a_2, c)\}$, $K_{4,1,1,1,1}$-$\{(a_1, b), (a_2, b)\}$, $K_{4,1,1,1,1}$-$\{(a_1, b), (a_2, c)\}$, $K_{3,2,1,1,1}$-$\{(a_1, b_1), (a_2, b_2)\}$, $K_{2,2,1,1,1}$-$\{(a_1, c), (b_1, c)\}$, $K_{2,2,1,1,1}$-$\{(b_1, c), (b_2, d)\}$, $K_{3,2,2,1}$-$\{(a_1, b_1), (a_1, c_1)\}$, $K_{3,2,2,1}$-$\{(a_1, b_1), (a_2, b_2)\}$, $K_{3,3,1,1}$-$\{(a_1, b_1), (a_2, b_2)\}$, $K_{4,2,1,1}$-$\{(a_1, b_1), (a_2, b_2)\}$, $K_{4,2,1,1}$-$\{(a_1, c), (a_2, c)\}$, $K_{4,2,1,1}$-$\{(a_1, b_1), (a_2, b_1)\}$, $K_{4,2,1,1}$-$\{(a_1, b_1), (a_2, c)\}$, $K_{4,3,1}$-$\{(a_1, b_1), (a_2, b_1)\}$, and $K_{4,3,1}$-$\{(a_1, b_1), (a_2, b_2)\}$ all have 1 vertex connected to every other. In each of these, if we delete that vertex we get a planar graph; so, by lemma 2, none are intrinsically linked.

$K_{3,1,1,1,1}$-$\{(b, c), (d, e)\}$ is equivalent to $K_{3,2,2}$, so it is not intrinsically linked.
$K_{4,2,1,1}$-$\{(a_1, c), (a_2, d)\}$ is a minor of $K_{4,1,1,1,1}$-$\{(a_1, b), (a_2, c)\}$ by lemma 1; simply combine parts D and E.
$K_{3,2,1,1}$-$\{(b_1, c), (b_2, d)\}$ is a minor of $K_{3,1,1,1,1}$-$\{(b, c), (d, e)\}$ by lemma 1; simply combine parts C and D.
$K_{4,2,2}$-$\{(a_1, b_1), (a_2, b_2)\}$, $K_{4,2,2}$-$\{(a_1, b_1), (a_2, b_1)\}$, and $K_{4,2,2}$-$\{(a_1, b_1), (a_2, c_1)\}$, are minors of $K_{4,2,1,1}$-$\{(a_1, b_1), (a_2, b_2)\}$, $K_{4,2,1,1}$-$\{(a_1, b_1), (a_2, b_1)\}$, and $K_{4,2,1,1}$-$\{(a_1, b_1), (a_2, c)\}$ respectively by lemma 1.
$K_{4,2,2} - \{(a_1, b_1), (a_1, c_1)\}$ is a minor of $K_{3,2,2,1} - \{(a_1, b_1), (a_1, c_1)\}$ and is, therefore, not intrinsically linked.
$K_{3,3,2}$-$\{(a_1, b_1), (a_2, b_2)\}$ is a minor of $K_{3,3,1,1}$-$\{(a_1, b_1), (a_2, b_2)\}$.
$K_{3,3,2}$-$\{(b_1, c_1), (b_2, c_2)\}$ is a minor of $K_{3,2,2,1}$-$\{(a_1, b_1), (a_2, b_2)\}$.
$K_{3,3,2}$-$\{(a_1, b_1), (b_1, c_1)\}$ is not intrinsically linked by the corollary to lemma 2; if you delete vertex a_2, the result is a planar graph.

$K_{4,3,1}$-$\{(a_1,c),(a_2,c)\}$ and $K_{4,3,1}$-$\{(a_1,b_1),(a_2,c)\}$ are minors of $K_{4,2,1,1}$-$\{(a_1,c),(a_2,c)\}$ and $K_{4,2,1,1}$-$\{(a_1,b_1),(a_2,c)\}$ respectively by lemma 1, and therefore are not intrinsically linked.

$K_{4,4}$-2e is a minor of $K_{4,4}$-e, which is minor minimally intrinsically linked, so it is not intrinsically linked. □

k	1	2	3	4	5
knotted	8-2e	5,5-2e	3,3,3-$\{(a_1,b_1),(a_1,b_2)\}$ 3,3,3-$\{(a_1,b_1),(b_2,c_1)\}$ 4,4,1-$\{(a_1,c),(b_1,c)\}$ 4,4,1-$\{(a_1,b_1),(b_1,c)\}$ 4,3,2-$\{(a_1,b_1),(a_1,b_2)\}$ 4,3,2-$\{(a_1,c_1),(b_1,c_2)\}$ 4,3,2-$\{(a_1,c_1),(b_1,c_1)\}$ 4,3,2-$\{(b_1,c_1),(b_2,c_1)\}$ 4,3,2-$\{(a_1,c_1),(a_1,c_2)\}$ 4,3,2-$\{(a_1,b_1),(b_2,c_1)\}$ 4,3,2-$\{(b_1,c_1),(b_1,c_2)\}$ 5,4,1-2e 5,3,2-2e 4,3,3-2e 4,4,2-2e	3,2,2,2-$\{(b,c),e\}$ 3,2,2,2-$\{(a_1,b_1),(a_1,b_2)\}$ 3,2,2,2-$\{(a_1,b_1),(a_2,b_1)\}$ 3,2,2,2-$\{(a_1,b_1),(a_2,c_1)\}$ 4,2,2,1-$\{(b,c),e\}$ 4,2,2,1-$\{(c,d),e\}$ 4,2,2,1-$\{(a_1,b_1),(a_1,b_2)\}$ 4,2,2,1-$\{(a_1,b_1),(a_1,d)\}$ 3,3,2,1-$\{(c,d),e\}$ 3,3,2,1-$\{(a,d),e\}$ 3,3,2,1-$\{(a_1,b_1),(a_1,b_2)\}$ 3,3,2,1-$\{(b_1,c_1),(b_1,c_2)\}$ 3,3,2,1-$\{(a_1,c_1),(b_1,c_1)\}$ 3,3,2,1-$\{(a_1,b_1),(a_2,c_1)\}$ 3,3,2,1-$\{(a_1,c_1),(a_2,c_1)\}$ 3,3,2,1-$\{(a_1,c_1),(b_1,c_2)\}$ 4,3,1,1-$\{(b,c),e\}$ 4,3,1,1-$\{(c,d),e\}$ 4,3,1,1-$\{(a_1,b_1),(a_1,b_2)\}$ 4,3,1,1-$\{(a_1,b_1),(a_1,c)\}$ 4,3,1,1-$\{(a_1,c),(a_1,d)\}$ 4,2,2,2-2e 3,3,2,2-2e 5,2,2,1-2e 4,3,2,1-2e 3,3,3,1-2e 5,3,1,1-2e 4,4,1,1-2e	3,2,1,1,1-$\{(b_1,c),(b_1,d)\}$ 3,2,1,1,1-$\{(b_1,c),(b_2,c)\}$ 4,2,1,1,1-$\{(c,d),e\}$ 4,2,1,1,1-$\{(b,c),e\}$ 4,2,1,1,1-$\{(a_1,b_1),(a_1,c)\}$ 4,2,1,1,1-$\{(a_1,b_1),(a_1,b_2)\}$ 4,2,1,1,1-$\{(a_1,c),(a_1,d)\}$ 3,3,1,1,1-$\{(b,c),e\}$ 3,3,1,1,1-$\{(c,d),e\}$ 3,3,1,1,1-$\{(a_1,b_1),(a_2,b_1)\}$ 3,2,2,1,1-$\{(a,d),e\}$ 3,2,2,1,1-$\{(c,d),e\}$ 3,2,2,1,1-$\{(b,c),e\}$ 3,2,2,1,1-$\{(d,e),e\}$ 3,2,2,1,1-$\{(a_1,b_1),(a_1,b_2)\}$ 3,2,2,1,1-$\{(a_1,b_1),(a_2,b_1)\}$ 3,2,2,1,1-$\{(a_1,b_1),(a_2,c_1)\}$ 2,2,2,2,1-2e 5,2,1,1,1-2e 4,3,1,1,1-2e 4,2,2,1,1-2e 3,3,2,1,1-2e
not knotted	7-2e	n,4-2e	3,3,3-$\{(a_1,b_1),(b_1,c_1)\}$ 3,3,3-$\{(a_1,b_1),(a_2,b_2)\}$ 4,4,1-$\{(a_1,b_1),(a_1,b_2)\}$ 4,4,1-$\{(a_1,b_1),(a_2,b_2)\}$ 4,4,1-$\{(a_1,c),(a_2,c)\}$ 4,4,1-$\{(a_1,b_1),(a_2,c)\}$ 4,3,2-$\{(a_1,b_1),(a_2,b_1)\}$ 4,3,2-$\{(a_1,b_1),(a_2,b_2)\}$ 4,3,2-$\{(a_1,c_1),(a_2,c_1)\}$ 4,3,2-$\{(a_1,b_1),(b_1,c_1)\}$ 4,3,2-$\{(a_1,b_1),(a_2,c_1)\}$ 4,3,2-$\{(b_1,c_1),(b_2,c_2)\}$ 4,3,2-$\{(a_1,c_1),(a_2,c_2)\}$ 4,3,2-$\{(a_1,b_1),(a_1,c_1)\}$ 3,3,2-2e n,2,2-2e n,3,1-2e	3,2,2,2-$\{(a_1,b_1),(a_1,c_1)\}$ 3,2,2,2-$\{(a_1,b_1),(a_2,b_2)\}$ 4,2,2,1-$\{(a_1,b_1),(a_2,b_1)\}$ 4,2,2,1-$\{(a_1,d),(a_2,d)\}$ 4,2,2,1-$\{(a_1,b_1),(a_2,b_2)\}$ 4,2,2,1-$\{(a_1,b_1),(a_2,c_1)\}$ 4,2,2,1-$\{(a_1,b_1),(a_2,d)\}$ 4,2,2,1-$\{(a_1,b_1),(a_1,c_1)\}$ 3,3,2,1-$\{(a_1,b_1),(b_1,c_1)\}$ 3,3,2,1-$\{(a_1,b_1),(a_2,b_2)\}$ 3,3,2,1-$\{(b_1,c_1),(b_2,c_2)\}$ 4,3,1,1-$\{(a_1,b_1),(a_2,b_1)\}$ 4,3,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 4,3,1,1-$\{(a_1,b_1),(a_2,c)\}$ 4,3,1,1-$\{(a_1,c),(a_2,d)\}$ 4,3,1,1-$\{(a_1,c),(a_2,c)\}$ 3,3,1,1-2e 2,2,2,2-2e 3,2,2,1-2e n,2,1,1-2e	3,2,1,1,1-$\{(a,b),e\}$ 3,2,1,1,1-$\{(a,c),e\}$ 3,2,1,1,1-$\{(c,d),e\}$ 3,2,1,1,1-$\{(b_1,c),(b_2,d)\}$ 4,2,1,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 4,2,1,1,1-$\{(a_1,b_1),(a_2,c)\}$ 4,2,1,1,1-$\{(a_1,b_1),(a_2,b_1)\}$ 4,2,1,1,1-$\{(a_1,c),(a_2,c)\}$ 4,2,1,1,1-$\{(a_1,c),(a_2,d)\}$ 3,3,1,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 3,2,2,1,1-$\{(a_1,b_1),(a_1,c_1)\}$ 3,2,2,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 2,2,2,1,1-2e n,1,1,1,1-2e

TABLE 7. Intrinsic Knotting of 2 Deficient Graphs

k	6	7	≥8
knotted	2,2,1,1,1,1-$\{(b_1,c),(b_2,c)\}$	2,1,1,1,1,1,1-$\{(a_1,b),(a_1,c)\}$	all
	2,2,1,1,1,1-$\{(b_1,c),(b_1,d)\}$	2,1,1,1,1,1,1-$\{(a_1,b),(a_2,b)\}$	
	2,2,1,1,1,1-$\{(a_1,d),(b_1,c)\}$	2,1,1,1,1,1,1-$\{(a_1,b),(c,d)\}$	
	3,1,1,1,1,1-$\{(b,c),(c,d)\}$	2,1,1,1,1,1,1-$\{(b,c),(c,d)\}$	
	4,1,1,1,1,1-$\{(b,c),e\}$	3,1,1,1,1,1,1-2e	
	4,1,1,1,1,1-$\{(a_1,b),(a_1,c)\}$	2,2,1,1,1,1,1-2e	
	3,2,1,1,1,1-$\{(a,c),e\}$		
	3,2,1,1,1,1-$\{(b,c),e\}$		
	3,2,1,1,1,1-$\{(c,d),e\}$		
	3,2,1,1,1,1-$\{(a_1,b_1),(a_1,b_2)\}$		
	3,2,1,1,1,1-$\{(a_1,b_1),(a_2,b_1)\}$		
	2,2,2,1,1,1-2e		
	5,1,1,1,1,1-2e		
	4,2,1,1,1,1-2e		
	3,3,1,1,1,1-2e		
not knotted	2,2,1,1,1,1-$\{(a,b),e\}$	2,1,1,1,1,1,1-$\{(a_1,b),(b,c)\}$	none
	2,2,1,1,1,1-$\{(c,d),e\}$	2,1,1,1,1,1,1-$\{(b,c),(d,e)\}$	
	2,2,1,1,1,1-$\{(b_1,c),(b_2,d)\}$	2,1,1,1,1,1,1-$\{(a_1,b),(a_2,c)\}$	
	2,2,1,1,1,1-$\{(a_1,c),(b_1,c)\}$	1,1,1,1,1,1,1-2e	
	3,1,1,1,1,1-$\{(a,b),e\}$		
	3,1,1,1,1,1-$\{(b,c),(d,e)\}$		
	4,1,1,1,1,1-$\{(a_1,b),(a_2,b)\}$		
	4,1,1,1,1,1-$\{(a_1,b),(a_2,c)\}$		
	3,2,1,1,1,1-$\{(a_1,b_1),(a_2,b_2)\}$		
	2,1,1,1,1,1-2e		

TABLE 8. Intrinsic Knotting of 2 Deficient Graphs (cont.)

5.2. **Intrinsic knotting.**

Theorem 12. *The 2-deficient graphs are classified with respect to intrinsic knotting according to Tables 7 and 8.*

Proof:
Knotted:

K_8-2e has 2 cases, K_8-$\{(a,b),(b,c)\}$ and K_8-$\{(a,b),(c,d)\}$. In the first case, delete vertex b to get K_7; in the second, contract edge (a,c) to get K_7. So K_8-2e is intrinsically knotted. Note that any 2-deficient graph with 8 or more parts will have K_8-2e as a minor.

$K_{5,5}$-2e has 2 cases. As shown in figure 8, by splitting vertex v and then adding the dashed lines, it can be seen that, in either case, $K_{5,5}$-2e has H_9 (from [6]) as a minor.

By lemma 1, $K_{5,4,1}$-2e, $K_{5,3,2}$-2e, $K_{3,3,2,2}$-2e, $K_{5,2,2,1}$-2e, $K_{4,3,2,1}$-2e, $K_{5,3,1,1}$-2e, $K_{4,4,1,1}$-2e, $K_{5,2,1,1,1}$-2e, $K_{4,3,1,1,1}$-2e, $K_{4,2,2,1,1}$-2e, $K_{3,3,2,1,1}$-2e, $K_{5,1,1,1,1,1}$-2e, $K_{4,2,1,1,1,1}$-2e, and $K_{3,3,1,1,1,1}$-2e all contain $K_{5,5}$-2e as a minor and are therefore intrinsically knotted.

$K_{4,3,3}$-2e is intrinsically knotted. In all cases, there is one removed edge connected to vertex b_1; delete that vertex to get $K_{4,3,2}$-e.

$K_{3,3,3,1}$-2e has $K_{4,3,3}$-2e as a minor by lemma 1, so it is intrinsically knotted.

$K_{4,4,2}$-2e is intrinsically knotted. In all cases, there is one removed edge connected to vertex a_1; delete vertex a_1 to get $K_{4,3,2}$-e.

By lemma 1, $K_{4,2,2,2}$-2e has $K_{4,4,2}$-2e as a minor.

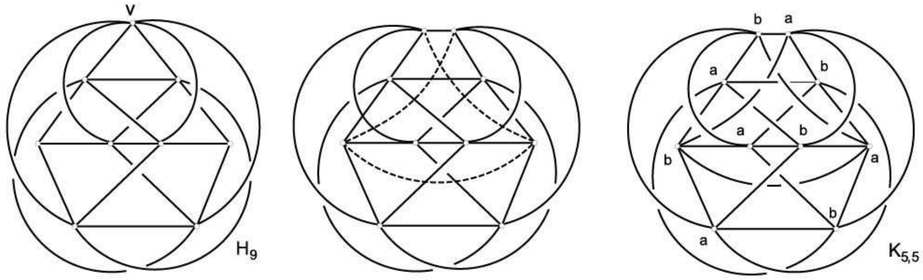

FIGURE 8. H_9 is a Minor of $K_{5,5}$-2e

$K_{2,2,2,2,1}$-2e is intrinsically knotted. In all cases, there will be an edge missing between A and B or between D and E. If we combine parts A and B in the first case, we get $K_{4,2,2,1}$-e or $K_{4,2,2,1}$, and if we combine parts D and E in the second case, we get $K_{3,2,2,2}$-e or $K_{3,2,2,2}$. So, in either case, $K_{2,2,2,2,1}$-2e has an intrinsically knotted minor.

By lemma 1, $K_{2,2,2,1,1,1}$-2e and $K_{2,2,1,1,1,1,1}$-2e have $K_{2,2,2,2,1}$-2e as a minor.

$K_{3,1,1,1,1,1,1}$-2e is intrinsically knotted. There is either an edge missing between parts A and B or between parts B and C. In the first case, we can combine A and B to get $K_{4,1,1,1,1,1}$-e or $K_{4,1,1,1,1,1}$, and in the second, we can combine parts B and C to get $K_{3,2,1,1,1,1}$-e or $K_{3,2,1,1,1,1}$.

By lemma 1, $K_{3,2,2,2}$-$\{(b,c),e\}$, $K_{4,2,2,1}$-$\{(c,d),e\}$, $K_{3,3,2,1}$-$\{(a,d),e\}$, $K_{4,3,1,1}$-$\{(c,d),e\}$, $K_{4,2,1,1,1}$-$\{(c,d),e\}$, $K_{4,2,1,1,1}$-$\{(b,c),e\}$, $K_{3,3,1,1,1}$-$\{(b,c),e\}$, $K_{3,3,1,1,1}$-$\{(c,d),e\}$, $K_{3,2,2,1,1}$-$\{(a,d),e\}$, $K_{3,2,2,1,1}$-$\{(c,d),e\}$, $K_{3,2,2,1,1}$-$\{(b,c),e\}$, $K_{3,2,2,1,1}$-$\{(d,e),e\}$, $K_{4,1,1,1,1,1}$-$\{(b,c),e\}$, $K_{3,2,1,1,1,1}$-$\{(a,c),e\}$, $K_{3,2,1,1,1,1}$-$\{(b,c),e\}$, $K_{3,2,1,1,1,1}$-$\{(c,d),e\}$, all have $K_{4,3,2}$-e as a minor, and are therefore all intrinsically knotted.

By lemma 1, $K_{4,2,2,1}$-$\{(b,c),e\}$ and $K_{4,3,1,1}$-$\{(b,c),e\}$ have $K_{4,4,1}$-e as a minor.

$K_{3,3,2,1}$-$\{(c,d),e\}$ has $K_{3,3,3}-e$ as a minor by lemma 1.

As shown in figure 3, H_9 is a minor of $K_{3,3,3}$. If we simply add fewer edges, we see that $K_{3,3,3}$-$\{(a_1,b_1),(a_1,b_2)\}$ and $K_{3,3,3}$-$\{(a_1,b_1),(b_2,c_1)\}$ are intrinsically knotted.

$K_{3,3,2,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,3,2,1}$-$\{(b_1,c_1),(b_1,c_2)\}$, $K_{3,3,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{3,2,2,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,2,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{3,2,1,1,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,1,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, and $K_{3,3,2,1}$-$\{(a_1,c_1),(a_2,c_1)\}$ have $K_{3,3,3}$-$\{(a_1,b_1),(a_1,b_2)\}$ as a minor by lemma 1.

$K_{3,2,2,1,1}$-$\{(a_1,b_1),(a_2,c_1)\}$, $K_{3,3,2,1}$-$\{(a_1,b_1),(a_2,c_1)\}$, and $K_{3,3,2,1}$-$\{(a_1,c_1),(b_1,c_2)\}$ have $K_{3,3,3}$-$\{(a_1,b_1),(b_2,c_1)\}$ as a minor by lemma 1.

As shown in figure 6, H_9 is a minor of $K_{4,4,1}$. If we simply add fewer edges, we can see that $K_{4,4,1}$-$\{(a_1,c),(b_1,c)\}$ and $K_{4,4,1}$-$\{(a_1,b_1),(b_1,c)\}$ are intrinsically knotted.

$K_{4,2,2,1}$-$\{(a_1,b_1),(a_1,d)\}$, $K_{4,3,1,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{4,2,1,1,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{4,2,1,1,1}$-$\{(a_1,c),(a_1,d)\}$, $K_{4,1,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$, and $K_{4,3,1,1}$-$\{(a_1,c),(a_1,d)\}$ have $K_{4,4,1}$-$\{(a_1,b_1),(b_1,c)\}$ as a minor by lemma 1.

H_9 is a minor of $K_{4,3,2}$ as shown in figure 5. If we add 2 fewer edges, we can see that $K_{4,3,2}$-$\{(a_1,c_1),(b_1,c_2)\}$, $K_{4,3,2}$-$\{(a_1,c_1),(b_1,c_1)\}$, $K_{4,3,2}$-$\{(b_1,c_1),(b_2,c_1)\}$, and $K_{4,3,2}$-$\{(a_1,c_1),(a_1,c_2)\}$ have H_9 as a minor, and are therefore intrinsically knotted.

As shown in figure 4, $K_{3,3,1,1}$ is a minor of $K_{4,3,2}$. If we add 2 fewer edges, we can see that $K_{4,3,2}$-$\{(a_1,b_1),(a_1,b_2)\}$ has $K_{3,3,1,1}$ as a minor, so it is intrinsically knotted.

$K_{4,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$ is intrinsically knotted; if we contract edge (a_1,c_1) we get $K_{3,3,1,1}$.

$K_{4,3,2}$-$\{(b_1,c_1),(b_1,c_2)\}$ is intrinsically knotted; it has B_9 from table 2 as a minor.

$K_{3,2,2,2}$-$\{(a_1,b_1),(a_1,b_2)\}$ has $K_{4,3,2}$-$\{(b_1,c_1),(b_1,c_2)\}$ as a minor by lemma 1.

$K_{3,2,2,2}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{4,2,2,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{4,3,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, and $K_{4,2,1,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$ have $K_{4,3,2}$-$\{(a_1,b_1),(a_1,b_2)\}$ as a minor by lemma 1.

$K_{3,3,2,1}$-$\{(a_1,c_1),(b_1,c_1)\}$ has $K_{4,3,2}$-$\{(a_1,c_1),(b_1,c_1)\}$ as a minor by lemma 1.

$K_{3,2,2,2}$-$\{(a_1,b_1),(a_2,c_1)\}$ has $K_{4,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$ as a minor by lemma 1.

$K_{3,2,1,1,1}$-$\{(b_1,c),(b_2,c)\}$ is equivalent to $K_{3,3,1,1}$.

$K_{3,2,1,1,1}$-$\{(b_1,c),(b_1,d)\}$ has H_8 from [6] as a minor.

$K_{2,2,1,1,1,1}$-$\{(b_1,c),(b_1,d)\}$, $K_{3,1,1,1,1,1}$-$\{(b,c),(c,d)\}$, and $K_{2,1,1,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$ have $K_{3,2,1,1,1}$-$\{(b_1,c),(b_1,d)\}$ as a minor by lemma 1.

$K_{2,2,1,1,1,1}$-$\{(b_1,c),(b_2,c)\}$, $K_{2,1,1,1,1,1,1}$-$\{(a_1,b),(a_2,b)\}$, and $K_{2,1,1,1,1,1,1}$-$\{(b,c),(c,d)\}$ have $K_{3,2,1,1,1}$-$\{(b_1,c),(b_2,c)\}$ as a minor by lemma 1.

$K_{2,2,1,1,1,1}$-$\{(a_1,d),(b_1,c)\}$ has $K_{3,3,1,1}$ as a minor.

$K_{2,1,1,1,1,1,1}$-$\{(a_1,b),(c,d)\}$ has 2 vertices connected to every other, if you delete those, you get a non-planar graph, so by lemma 4, it is intrinsically knotted.

Not Knotted:

Any graph that was not knotted with 1 edge removed will clearly not be knotted with 2 edges removed. The following graphs fit that description: K_7-2e ($= K_{1,1,1,1,1,1,1}$-2e), $K_{n,4}$-2e, $K_{3,3,2}$-2e, $K_{n,2,2}$-2e, $K_{n,3,1}$-2e, $K_{3,3,1,1}$-2e, $K_{2,2,2,2}$-2e, $K_{3,2,2,1}$-2e, $K_{n,2,1,1}$-2e, $K_{3,2,1,1,1}$-$\{(a,b),e\}$, $K_{3,2,1,1,1}$-$\{(a,c),e\}$, $K_{3,2,1,1,1}$-$\{(c,d),e\}$, $K_{2,2,2,1,1}$-2e, $K_{n,1,1,1,1}$-2e, $K_{2,2,1,1,1,1}$-$\{(a,b),e\}$, $K_{2,2,1,1,1,1}$-$\{(c,d),e\}$, $K_{3,1,1,1,1,1}$-$\{(a,b),e\}$, and $K_{2,1,1,1,1,1}$-2e.

If a graph has 2 vertices connected to every other vertex and to one another and the deletion of those vertices results in a planar graph, lemma 4 states that the original graph is not intrinsically knotted. The following graphs are of that form: $K_{4,3,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{4,3,1,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{4,2,1,1,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{4,2,1,1,1}$-$\{(a_1,b_1),(a_2,c)\}$, $K_{4,2,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{4,2,1,1,1}$-$\{(a_1,c),(a_2,c)\}$, $K_{3,3,1,1,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{3,2,2,1,1}$-$\{(a_1,b_1),(a_1,c_1)\}$, $K_{3,2,2,1,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{2,2,1,1,1,1}$-$\{(b_1,c),(b_2,d)\}$, $K_{2,2,1,1,1,1}$-$\{(a_1,c),(b_1,c)\}$, $K_{4,1,1,1,1,1}$-$\{(a_1,b),(a_2,b)\}$, $K_{4,1,1,1,1,1}$-$\{(a_1,b),(a_2,c)\}$, $K_{3,2,1,1,1,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{2,1,1,1,1,1,1}$-$\{(a_1,b),(b,c)\}$, $K_{2,1,1,1,1,1,1}$-$\{(b,c),(d,e)\}$, and $K_{2,1,1,1,1,1,1}$-$\{(a_1,b),(a_2,c)\}$.

$K_{3,1,1,1,1,1}$-$\{(b,c),(d,e)\}$ and $K_{3,2,1,1,1}$-$\{(b_1,c),(b_2,d)\}$ are minors of $K_{2,1,1,1,1,1,1}$-$\{(b,c),(d,e)\}$.

$K_{4,2,1,1,1}$-$\{(a_1,c),(a_2,d)\}$, $K_{4,3,1,1}$-$\{(a_1,c),(a_2,d)\}$, $K_{4,3,1,1}$-$\{(a_1,b_1),(a_2,c)\}$, $K_{4,2,2,1}$-$\{(a_1,b_1),(a_2,d)\}$, $K_{4,2,2,1}$-$\{(a_1,b_1),(a_2,c_1)\}$, $K_{4,2,2,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{4,3,2}$-$\{(a_1,c_1),(a_2,c_2)\}$, $K_{4,3,2}$-$\{(a_1,b_1),(a_2,c_1)\}$, $K_{4,3,2}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{4,4,1}$-$\{(a_1,b_1),(a_2,c)\}$, and $K_{4,4,1}$-$\{(a_1,b_1),(a_2,b_2)\}$ are minors of $K_{4,1,1,1,1,1}$-$\{(a_1,b),(a_2,c)\}$.

$K_{4,3,1,1}$-$\{(a_1,c),(a_2,c)\}$, $K_{4,2,2,1}$-$\{(a_1,d),(a_2,d)\}$, $K_{4,2,2,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{4,3,2}$-$\{(a_1,c_1),(a_2,c_1)\}$, $K_{4,3,2}$-$\{(a_1,b_1),(a_2,b_1)\}$, and $K_{4,4,1}$-$\{(a_1,c),(a_2,c)\}$ are minors of $K_{4,1,1,1,1,1}$-$\{(a_1,b),(a_2,b)\}$.

$K_{3,3,2,1}$-$\{(b_1,c_1),(b_2,c_2)\}$, $K_{3,3,2,1}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{3,2,2,2}$-$\{(a_1,b_1),(a_2,b_2)\}$, $K_{4,3,2}$-$\{(b_1,c_1),(b_2,c_2)\}$ and $K_{3,3,3}$-$\{(a_1,b_1),(a_2,b_2)\}$ are minors of $K_{3,2,1,1,1,1}$-$\{(a_1,b_1),(a_2,b_2)\}$.

$K_{3,3,2,1}$-$\{(a_1,b_1),(b_1,c_1)\}$, $K_{3,2,2,2}$-$\{(a_1,b_1),(a_1,c_1)\}$, $K_{4,3,2}$-$\{(a_1,b_1),(a_1,c_1)\}$, $K_{4,3,2}$-$\{(a_1,b_1),(b_1,c_1)\}$, $K_{3,3,3}$-$\{(a_1,b_1),(b_1,c_1)\}$, and $K_{4,2,2,1}$-$\{(a_1,b_1),(a_1,c_1)\}$ are minors of $K_{3,2,2,1,1}$-$\{(a_1,b_1),(a_1,c_1)\}$.

$K_{4,4,1}$-$\{(a_1,b_1),(a_1,b_2)\}$ is not intrinsically knotted by the corollary to lemma 4. Delete vertices a_2 and c for a planar graph. □

5.3. **Proof of Adams' conjecture for 2-deficient graphs.**

Theorem 13. *If G is a 2-deficient graph, and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked.*

Proof:

It suffices to verify the theorem for minimal examples of 2-deficient graphs.

- $k = 1$**:** K_8-2e is intrinsically knotted; if we remove a vertex, we get K_7-2e, K_7-e, or K_7, all of which are intrinsically linked.
- $k = 2$**:** $K_{5,5}$-2e is intrinsically knotted; if we remove a vertex we get $K_{5,4}$-2e, $K_{5,4}$-e, or $K_{5,4}$, all of which are intrinsically linked.

$k=3$**:** If a vertex is removed from a minimal knotted 2-deficient graph that has 3 parts, the result is one of the following graphs: $K_{4,3,1}$-$\{(a_1,c),(b_1,c)\}$, $K_{4,3,1}$-$\{(a_1,b_1),(b_1,c)\}$, $K_{4,3,1}$-$\{(b_1,c),(b_2,c)\}$, $K_{4,3,1}$-$\{(a_1,b_1),(b_2,c)\}$, $K_{4,3,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{4,3,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{3,3,2}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(a_2,c_1)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(b_1,c_1)\}$, $K_{3,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(b_1,c_2)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(a_1,c_2)\}$, $K_{4,2,2}$-$\{(a_1,b_1),(b_2,c_1)\}$, $K_{4,2,2}$-$\{(a_1,b_1),(b_1,c_1)\}$, $K_{4,2,2}$-$\{(b_1,c_1),(b_2,c_1)\}$, $K_{4,2,2}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{5,3,1}$-2e, $K_{4,4,1}$-2e, $K_{4,3,2}$-2e, $K_{3,3,3}$-2e, $K_{5,2,2}$-2e, $K_{5,4}$-2e, $K_{4,4}$-e, or one of these graphs with 1 or 2 fewer edges missing. Note that they are all intrinsically linked.

$k=4$**:** If a vertex is removed from a minmal knotted 2-deficient graph that has 4 parts, the result is one of the following graphs: $K_{4,3,1}$-$\{(b,c),e\}$, $K_{4,3,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{4,3,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{3,3,2}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(a_2,c_1)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(b_1,c_1)\}$, $K_{3,3,2}$-$\{(a_1,b_1),(b_2,c_1)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(b_1,c_2)\}$, $K_{3,3,2}$-$\{(a_1,c_1),(a_1,c_2)\}$, $K_{4,2,2}$-$\{(b,c),e\}$, $K_{4,2,2}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{5,3,1}$-2e, $K_{4,4,1}$-2e, $K_{4,3,2}$-2e, $K_{3,3,3}$-2e, $K_{5,2,2}$-2e, $K_{4,2,1,1}$-$\{(b,c),e\}$, $K_{4,2,1,1}$-$\{(c,d),e\}$, $K_{4,2,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{4,2,1,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{4,2,1,1}$-$\{(a_1,c),(a_1,d)\}$, $K_{3,3,1,1}$-$\{(b,c),e\}$, $K_{3,3,1,1}$-$\{(c,d),e\}$, $K_{3,3,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,2,1}$-$\{(b,c),e\}$, $K_{3,2,2,1}$-$\{(c,d),e\}$, $K_{3,2,2,1}$-$\{(a,d),e\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_2,c_1)\}$, $K_{2,2,2,2}$-2e, $K_{5,2,1,1}$-2e, $K_{4,3,1,1}$-2e, $K_{4,2,2,1}$-2e, $K_{3,3,2,1}$-2e, $K_{3,2,2,2}$-2e, or one of these graphs with 1 or 2 fewer edges missing. Note that they are all intrinsically linked.

$k=5$**:** If a vertex is removed from a minimal knotted 2-deficient graph that has 5 parts, the result is one of the following graphs: $K_{3,2,1,1}$-$\{(b_1,c),(b_1,d)\}$, $K_{3,2,1,1}$-$\{(b_1,c),(b_2,c)\}$, $K_{4,2,1,1}$-$\{(b,c),e\}$, $K_{4,2,1,1}$-$\{(c,d),e\}$, $K_{4,2,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{4,2,1,1}$-$\{(a_1,b_1),(a_1,c)\}$, $K_{4,2,1,1}$-$\{(a_1,c),(a_1,d)\}$, $K_{3,3,1,1}$-$\{(b,c),e\}$, $K_{3,3,1,1}$-$\{(c,d),e\}$, $K_{3,3,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,2,1}$-$\{(b,c),e\}$, $K_{3,2,2,1}$-$\{(c,d),e\}$, $K_{3,2,2,1}$-$\{(a,d),e\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{3,2,2,1}$-$\{(a_1,b_1),(a_2,c_1)\}$, $K_{2,2,2,2}$-2e, $K_{5,2,1,1}$-2e, $K_{4,3,1,1}$-2e, $K_{4,2,2,1}$-2e, $K_{3,3,2,1}$-2e, $K_{2,2,1,1,1}$-$\{(b_1,c),(b_2,c)\}$, $K_{2,2,1,1,1}$-$\{(b_1,c),(b_1,d)\}$, $K_{3,1,1,1,1}$-$\{(b,c),(c,d)\}$, $K_{4,1,1,1,1}$-$\{(b,c),e\}$, $K_{4,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$, $K_{3,2,1,1,1}$-$\{(a,c),e\}$, $K_{3,2,1,1,1}$-$\{(b,c),e\}$, $K_{3,2,1,1,1}$-$\{(c,d),e\}$, $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{2,2,2,1,1}$-2e, $K_{5,1,1,1,1}$-2e, $K_{4,2,1,1,1}$-2e, $K_{3,3,1,1,1}$-2e, $K_{3,2,2,1,1}$-2e, or one of these graphs with 1 or 2 fewer edges missing. Note that they are all intrinsically linked.

$k=6$**:** If a vertex is removed from a minimal knotted 2-deficient graph that has 6 parts, the result is one of the following graphs: $K_{2,2,1,1,1}$-$\{(b_1,c),(b_2,c)\}$, $K_{2,2,1,1,1}$-$\{(a_1,c),(b_1,d)\}$, $K_{2,2,1,1,1}$-$\{(b_1,c),(b_1,d)\}$, $K_{3,1,1,1,1}$-$\{(b,c),(c,d)\}$, $K_{4,1,1,1,1}$-$\{(b,c),e\}$, $K_{4,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$, $K_{3,2,1,1,1}$-$\{(a,c),e\}$, $K_{3,2,1,1,1}$-$\{(b,c),e\}$, $K_{3,2,1,1,1}$-$\{(c,d),e\}$, $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_1,b_2)\}$, $K_{3,2,1,1,1}$-$\{(a_1,b_1),(a_2,b_1)\}$, $K_{2,2,2,1,1}$-2e, $K_{5,1,1,1,1}$-2e, $K_{4,2,1,1,1}$-2e, $K_{3,3,1,1,1}$-2e, $K_{2,1,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$, $K_{2,1,1,1,1,1}$-$\{(a_1,b),(a_2,b)\}$, $K_{2,1,1,1,1,1}$-$\{(a_1,b),(c,d)\}$, $K_{2,1,1,1,1,1}$-$\{(b,c),(c,d)\}$, $K_{3,1,1,1,1,1}$-2e, $K_{2,2,1,1,1,1}$-2e, $K_{4,1,1,1,1,1}$-2e, $K_{3,2,1,1,1,1}$-2e, or one of these graphs with 1 or 2 fewer edges missing. Note that they are all intrinsically linked.

$k=7$**:** If a vertex is removed from a minimal knotted 2-deficient graph that has 7 parts, the result is one of the following graphs: $K_{2,1,1,1,1,1}$-$\{(a_1,b),(a_1,c)\}$, $K_{2,1,1,1,1,1}$-$\{(a_1,b),(a_2,b)\}$, $K_{2,1,1,1,1,1}$-$\{(a_1,b),(c,d)\}$, $K_{2,1,1,1,1,1}$-$\{(b,c),(c,d)\}$, $K_{3,1,1,1,1,1}$-2e, $K_{2,2,1,1,1,1}$-2e, $K_{1,1,1,1,1,1,1}$-2e, $K_{2,1,1,1,1,1,1}$-2e, or one of these graphs with 1 or 2 fewer edges missing. Note that they are all intrinsically linked.

$k\geq 8$**:** If a vertex is removed from a knotted 2-deficient graph that has 8 or more parts, the result is a 2-deficient, 1-deficient, or complete partite graph with 7 or more parts, all of which are intrinsically linked.

□

6. Graphs on 8 vertices

In this section we provide a classification of intrinsically knotted graphs on 8 vertices. We verify that removing a vertex from any of these results in an intrinsically linked graph. We also discuss a question of Sachs [10] about maximal unlinked and unknotted graphs.

6.1. **Classification.** Graphs on 8 vertices are subgraphs of K_8. We will examine in turn subgraphs which are obtained by removing 1, 2, 3, … edges from K_8. We have already noted that K_8, K_8-e, and both K_8-2e graphs are intrinsically knotted. Of the five graphs K_8-3e, the three which are intrinsically knotted can all

be obtained by removing two edges from $K_{2,1,1,1,1,1,1}$. (See classification of 2-deficient graphs above. Note that $2,1,1,1,1,1,1-\{(a,b),(c,d)\}$ and $2,1,1,1,1,1,1-\{(b,c),(c,d)\}$ are the same graph.)

There are 11 graphs of the form K_8-4e. Seven of these are not knotted as they can be realized by removing an edge from one of the two unknotted K_8-3e graphs. The remaining four are intrinsically knotted. Three of these four are of the form $K_{2,2,1,1,1,1}-2e$. The fourth is obtained by removing 4 edges all incident to the same vertex. This graph is intrinsically knotted as it contains K_7 as a minor.

There are 24 graphs K_8-5e. All but 4 of these are not knotted as they are minors of an unknotted K_8-4e. The four intrinsically knotted K_8-5e's are perhaps most easily described in terms of their complementary graphs. In figure 9, Graph i is $K_{3,2,1,1,1}-(b,c)$, or, equivalently, $K_{3,1,1,1,1,1}-\{(b,c),(c,d)\}$. The other three graphs in the

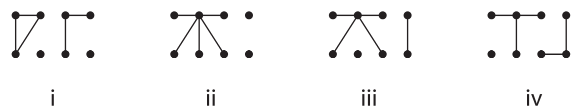

FIGURE 9. Intrinsically knotted K_8-5e's

figure have K_7 as a minor and are, therefore, intrinsically knotted.

Of the 56 K_8-6e graphs, all but 6 are minors of unknotted K_8-5e's. In figure 10, Graph i is $K_{3,2,1,1,1}-$

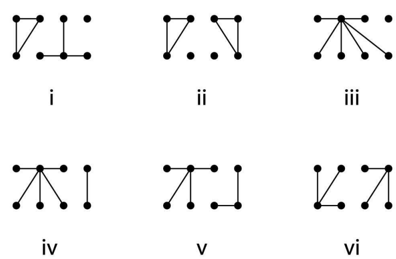

FIGURE 10. Intrinsically knotted K_8-6e's

$\{(b_1,c),(b_1,d)\}$ while Graph ii is $K_{3,3,1,1}$, or, equivalently, $K_{3,2,1,1,1}-\{(b_1,c),(b_2,c)\}$. The remaining four graphs are obtained by splitting a vertex of K_7 and are therefore intrinsically knotted.

Only 2 of the K_8-7e graphs are not minors of some unknotted K_8-6e. One of these two is H_8 [6]. The other is K_7 with one additional vertex. These are both intrinsically knotted. Moreover, K_7 and H_8 are minor minimal [6]. Thus any subgraph of K_8 obtained by removing 8 or more edges is not intrinsically knotted.

In total then, there are twenty intrinsically knotted graphs on 8 vertices.

6.2. **Proof of Adams' conjecture for graphs on** 8 **vertices.**

Theorem 14. *If G is an intrinsically knotted graph on 8 vertices and any one vertex and the edges coming into it are removed, the remaining graph is intrinsically linked.*

Proof: We have already verified this for the knotted graphs that are 0-, 1-, or 2-deficient complete partite graphs. Most of the other knotted graphs have K_7 as a minor. On removing a vertex, the resulting graph will have K_6 as a minor and be intrinsically linked. Foisy has shown that the removal of a vertex from H_8 results in an intrinsically linked graph [5].

□

6.3. **A question of Sachs.** Sachs [10] asked if a graph on n vertices that is not intrinsically linked could have more than $4n-10$ edges. Using lemma 4, we can ask a similar question about intrinsically knotted graphs. For $n \geq 5$, a planar triangulation with $n-2$ vertices will have $3(n-4)$ edges. Adding a K_2 gives a graph with $5n-15$ edges that is not intrinsically knotted by lemma 4. This is the maximum for $n = 5, 6, 7$, and our analysis of graphs on 8 vertices shows that it is also the maximum for $n = 8$. In other words, a graph with more than $5n-15$ edges on n vertices is intrinsically knotted when $5 \leq n \leq 8$. Is this also true for larger n?
Question: Is there a graph on n vertices that is not intrinsically knotted and has more than $5n-15$ edges?

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